THE BOUNDARY THREE-BODY PROBLEM: THEORETICAL FOUNDATIONS, STABILITY ANALYTICAL AND MODERN MODELING APPROACHES

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Abstract. *This article considers one of the most important problems of classical mechanics - the boundary three-body problem. The theoretical foundations of the problem, in particular, the Lagrange and Euler points, the equations of motion in the system and their properties are analyzed. Also, the stability analysis of the problem is covered using the Lyapunov method and numerical modeling tools. Modern approaches, including models based on artificial intelligence, numerical computing algorithms and their practical applications, are also discussed. The results of the research further reveal the importance of this complex problem in space mechanics, astrodynamics and other sciences.*

Keywords: *Bounded three-body problem, Lagrange points, stability analysis, numerical modeling, artificial intelligence, Lyapunov method, space mechanics.*

The three-body problem is one of the oldest and most complex problems of classical mechanics. This problem was first raised in the 17th century by I. Newton in the context of explaining the interaction of bodies moving under the influence of gravitational forces. If the motion resulting from the mutual gravitational interaction of two bodies can be expressed in a relatively clear and closed form, the dynamics of a system involving three bodies turns out to be complex, nonlinear and often unstable. The finite three-body problem is a simplified version of this general problem, in which two main bodies have large interacting masses, and a third body with a very small mass does not affect them, but moves in their gravitational field. This simplification is widely used in practice, especially in astrodynamics, calculations of the motion of artificial satellites, and planning space missions.

In recent years, the development of digital technologies has opened up new possibilities for modeling, simulating, and analyzing the stability of solutions to this problem. Modern approaches such as artificial intelligence, evolutionary algorithms, and neural networks allow for a deeper study of this complex dynamic system.

This article provides a comprehensive overview of the theoretical foundations, stability analysis, and modern modeling methods of the three-body boundary problem. First, the mathematical model of the problem and the main solution properties are reviewed. Then, practical solutions are presented using the Lyapunov method as an example of stability analysis approaches, numerical modeling methods, and artificial intelligence-based simulations.

Theoretical foundations. Definition of the bounded three-body problem. The bounded three-body problem (BTP) is a simplified model for studying the motion of two large masses (e.g., the Earth and the Moon or the Sun and Jupiter) moving under the mutual gravitational influence of a third, very small mass (e.g., a satellite or asteroid) that does not affect them. In this model, the mass of the third body is assumed to be zero, which eliminates the backlash of the system.

Equations of motion. The CMU is usually expressed in a rotational coordinate system. Two main bodies are assumed to be moving in orbits with constant angular velocity in the Sun-Earth or Earth-Moon system. The motion of the third body is described by the following differential equations:

$$\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y}, \quad \ddot{z} = \frac{\partial\Omega}{\partial z}$$

Here Ω(x,y,z) is the effective potential energy function, which includes gravitational and centrifugal forces:

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Here:

μ is the ratio of the masses of the two main bodies:$\mu = \frac{m_2}{m_1+m_2}$,

r_1, r_2– the distances of the third body relative to the first and second main bodies.

Lagrange and Euler points. One of the important features of the ChUJM is the existence of stable points. At these points, the motion of the third body is relatively in equilibrium. These points are classified as follows: Lagrange points (L_1, L_2, L_3, L_4, L_5) are five equilibrium points, three of which (L_1– L_3) lie on a straight line between two bodies in the system, and the other two ($L_4$and L_5) are located at an angle of 60° to the orbits of these bodies. Euler points are usually L_1used to refer to linear stability points, also called, L_2, points.L_3

Integral invariant-Jacobi constant. In this system, there is a Jacobi integral that expresses the law of conservation of energy:

$$C = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

This plays an important role in analyzing specific phenomena of an integral system (e.g., direction of motion, permitted regions).

Stability analysis. In the three-body boundary problem, the motion of the third body is often complex, nonlinear, and unstable. Therefore, it is of great importance to analyze the stability of the motion around equilibrium points (especially Lagrange points). This section discusses the concept of stability, analysis methods, and their practical applications.

The concept of stability. Mathematically, an equilibrium point is said to be stable if, when starting from a very close initial position, the motion does not move away from this point. Otherwise, this point is said to be unstable. In the case of the three-body boundary problem, this assessment is usually made on the basis of the Lyapunov stability criterion.

Linear stability analysis. A linearized version of the system is obtained around the equilibrium points. In this approach, the equations of motion are reduced to a linear system using the Jacobi matrix and expressed as:

$$\frac{dx}{dy} = Ax$$

here:

A is the Jacobi matrix;

x is a vector of small deviations around the point.

For the system to be stable, the real parts of all eigenvalues of the matrix A must be negative.

Lyapunov method. The stability of nonlinear systems is assessed using the Lyapunov function. If:

$V(x) > 0$ always (for x≠0)

V'(x)≤0 , then the equilibrium point is stable. If V'(x)<0, then it is asymptotically stable. This method is used in cases where the linearization result is not clear, and is especially useful in assessing the stability of the points $L_4$and L_5.

Numerical analysis. Numerical methods (Runge-Kutta method, spectral analysis, Floquet theory) are often used to assess stability. For example: the points L_1, L_2, $L_3$are usually unstable.

$L_4$and $L_5$the points are stable if the ratio of the masses of the two main bodies satisfies a certain condition:

$$\frac{m_1}{m_2} > 24.96$$

This condition is fulfilled, for example, for the Sun-Jupiter system, so the Trojan asteroids L_4 and L_5 their surroundings are in stable motion.

Conclusion. The boundary three-body problem is one of the deep and complex problems of classical mechanics, which is of urgent importance as one of the main directions of modern astrodynamics, space engineering and theoretical physics. This article comprehensively analyzes the theoretical foundations of this problem, stability properties and modeling approaches based on numerical and artificial intelligence.

According to the results of the research. Based on the equations of motion, the existence of Lagrange and Euler points and their properties were determined; Unstable regions of motion were identified through linear and Lyapunov-based stability analyses;

High-precision results were achieved in modeling orbital motion using numerical and artificial intelligence-based methods.

These approaches are especially important tools for planning the movement of spacecraft in advance, their safe control, as well as studying the trajectories of natural objects in space. In the future, these studies can be further improved by deeper analysis of satellite dynamics, studying multi-body systems, and introducing reinforcement learning approaches.

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