DEVELOPMENT OF A HYBRID SPECTRAL FRACTAL MODEL AND ALGORITHM FOR JEWELRY

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Abstract. *This article examines the development of a Hybrid Spectral Fractal Model (HSFM) for jewelry. The article describes the theoretical foundations and mathematical expression of the HSFM. The model combines three main components: parametric surfaces, fractal structures, and an implicit shell. Parametric surfaces provide the basic shape of the object, while fractal structures exhibit refined complexity and meet the requirements of implicit shell manufacturing. The model provides a complete solution for jewelry design, combining methods of differential geometry, fractal theory, and implicit modeling. Based on the model, the results obtained for various iteration n values are presented graphically.*

Keywords: *Fractal, jewelry, spectral model, FFT, Hybrid Spectral Fractal Model, density function (PDF)*

The following discusses the development of the Hybrid Spectral Fractal Model (HSFM) and its computational experiments. The HSFM algorithm is implemented through a sequence of six core stages [1,2]. Each stage involves a precise series of mathematical operations and builds upon the results of the preceding step. The algorithm operates on a discrete grid structure and leverages the efficiency of the Fast Fourier Transform (FFT).

In the first stage, the parametric surface $\mathrm{P}(u,v)$ is discretized using an $M \times N$ grid, and independent white noise values are generated at each grid point:

$$\eta_{m,n} \sim \mathcal{N}(0,1),\; m = 0,1,\ldots,M-1,\; n = 0,1,\ldots,N-1. \qquad (1)$$

where $\mathcal{N}(0,1)$ represents a standard normal distribution with zero mean and unit variance. The probability density function (PDF) of the normal distribution is defined as follows:

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right). \qquad (2)$$

Spectrally, white noise possesses equal power across all frequencies (a flat spectrum), making it an ideal starting material for spectral shaping [3,4].

In the second stage, the white noise field is transformed from the spatial domain to the frequency domain [5,6]. A two-dimensional Discrete Fourier Transform (2D DFT) is applied as follows:

$$\hat{\eta}_{k,l} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \eta_{m,n} \cdot \exp\left(-2\pi i \left(\frac{km}{M} - \frac{\ln}{N}\right)\right). \quad (3)$$

where $k = 0,1,...,M-1$ and $l = 0,1,...N-1$ - frequency indices, $\hat{\eta}_{k,l} \in \mathbb{C}$ - complex Fourier coefficient. $i = \sqrt{-1}$ - abstract unit.

Direct application of the Discrete Fourier Transform formula has a computational complexity of $O\left(M^2 N^2\right)$. Instead, the Fast Fourier Transform (FFT) algorithm is utilized [7,8]. The FFT reduces the computational complexity to $O\left[MN \log\left(MN\right)\right]$. The frequency coordinates are defined as follows:

$$\omega_u^{(k)} = \begin{cases} \frac{2\pi k}{M\Delta_u}, & k \le M/2 \\ \frac{2\pi\left(k-M\right)}{M\Delta_u}, & k > M/2 \end{cases}$$

$$\omega_v^{(l)} = \begin{cases} \frac{2\pi l}{N\Delta_v}, & l \le N/2 \\ \frac{2\pi\left(l-N\right)}{N\Delta_v}, & l > N/2 \end{cases}$$

where $\Delta_u = 1/M$ va $\Delta_v = 1/N$ - discretization steps in parametric space. $\omega_{\max} = \pi / \Delta$ - determines the maximum sampling frequency [9].

In the third stage, each frequency component is weighted via a spectral shaping filter. First, the radial frequency is calculated.

$$\omega_r^{(k,l)} = \sqrt{(\omega_u^{(k)})^2 + (\omega_v^{(l)})^2}. \quad (4)$$

Then, a spectral shaping filter is applied:

$$H_{k,l} = \frac{A}{\left((\omega_r^{(k,l)})^2 + \omega_{\min}^2\right)^{\beta/4}}, \quad (5)$$

where A - normalization constant, $\beta \in [2,4]$ - spectral slope, $\omega_r = 0.1 \cdot \omega_{\max}$ - minimum frequency limit.

The normalization constant is calculated according to Parseval's theorem [10]:

$$A = \left(\sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \frac{1}{\left((\omega_r^{(k,l)})^2 + \omega_{\min}^2 \right)^{\beta/2}} \right)^{-1/2}$$

Spectral formation is applied:

$$\hat{g}_{k,l} = H_{k,l} \cdot \hat{\eta}_{k,l} \qquad (6)$$

This is an element-by-element multiplication and the multiplication in the frequency domain is equivalent to the convolution in the spatial domain (the convolution theorem).

In the fourth stage, a technological low-permeability filter is used. This filter removes structures smaller than the minimum detail size of the selected manufacturing technology. Technological filter in Gaussian form:

$$T_{k,l} = \exp\left(-\frac{(\omega_r^{(k,l)})^2}{2\omega_{cut}^2} \right),$$

where is the cutting frequency: $\omega_{cut} = \frac{2\pi}{d_{\min}},$

$d_{\min}$ - the minimum detail size of a technology.

Technological filtering is applied:

$$\hat{f}_{k,l} = T_{k,l} \cdot \hat{g}_{k,l} \qquad (7)$$

In the fifth step, the filtered signal is returned to the spatial domain. The inverse discrete Fourier transform is applied:

$$f_{m,n} = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \hat{f}_{k,l} \cdot \exp\left(2\pi i \left(\frac{km}{M} + \frac{ln}{N} \right) \right).$$

The result of the inverse discrete Fourier transform can generally be a complex number. However, since the initial signal is real and symmetric filters are used, the complex part will be small at the level of computational error. Therefore, only the actual part is taken:

$$f_{m,n}^{(real)} = \mathrm{Re}\left\{ f_{m,n}\left[H_{k,l} \cdot T_{k,l} \cdot \hat{\eta} \right] \right\}. \qquad (8)$$

In practical programming, the inverse fast Fourier transform algorithm is used, and the computational complexity is $O\left[MN \log(MN) \right]$.

In the final stage, the obtained fractal field is applied to a parametric surface. The field will be normalized to the range $[0,1]$ first:

$$\tilde{f}_{m,n} = \frac{f_{m,n}^{real} - f_{\min}}{f_{\max} - f_{\min}}.$$

hare $f_{\min} = \min f_{m,n}^{(real)}$ and $f_{\max} = \max f_{m,n}^{(real)}$.

Saturation process restricts too large values:

$$f_{m,n}^{(sat)} = \max\left(t_{\min}, \min\left(t_{\max}, \tilde{f}_{m,n}\right)\right), \tag{9}$$

where $t_{\min} = 0.05$ and $t_{\max} = 0.95$ are saturation limits. This prevents very deep depressions or high peaks on the surface.

(9) – according to formulas (10) as follows [11, 12]:

$$S^{*}_{m,n} = S_{m,n}(u,v) + A_{m,n}(u,v) \cdot f_{m,n}^{(sat)} \cdot \hat{n}_{m,n}.$$

where: $S_{m,n} = S(u_m, v_n)$ - base surface, $A(u,v)$ - lokal amplituda, $\hat{n}_{m,n}$ - surface normal.

$$S^{*}(u,v) = S_t(u,v) + A_{\max} \cdot A_{\kappa}(u,v) \cdot A_{\text{area}}(u,v) \cdot A_{\text{ovr}}(u,v) \cdot \left(\max\left(t_{\min}, \min\left(t_{\max}, \tilde{f}_{m,n}\right)\right)\right) \quad (10^{*})$$

$$\cdot \hat{n}(u,v).$$

Formula (10*) represents the GSFM, which smoothly combines the embossed surface and the shell in accordance with technological requirements, providing a ready-made geometry for production.

a)	b)	c)	d)
$\beta = 2.2$, $d_{\min} = 0.1mm$ CNC, $h = 0.3mm$	$\beta = 3.0$, $d_{\min} = 0.2mm$,FDM, $h = 0.8mm$.	$\beta = 3.6$, $d_{\min} = 0.5mm$ quyma $h = 0.25mm$	$\beta = 3.2$, $d_{\min} =$ 0.3 mm, SLS, $h = 0.7$ mm.

Figure 1. Fractal rings developed based on GSFM

Correction to account for technological limitations.

$$\beta_e = \beta + \Delta\beta_{tech} \tag{11}$$

where * is the additional smoothness for low-precision technologies [13, 14].

Spectral accuracy verification:

$$\varepsilon_{\beta} = \frac{|\beta_t - \beta_e|}{\beta_t} \times 100\%$$

$\varepsilon_{\beta} < 2\%$ achieved for GSFM.

In this section, a complete numerical solution of the GSFM was developed and tested. A six-stage algorithm was described, consisting of white noise generation, rapid Fourier transform, spectral formation, technological filtering, IFFT, and parametric surface application. Samples of fractal jewelry were developed based on the GSFM algorithm (see Fig. 1).

Conclusion

In this section, a complete numerical solution of the GSFM model was developed and tested. An algorithm consisting of 6 stages is described: white noise generation, FFT, spectral formation, technological filtering, IFFT, and parametric surface application. The computational complexity is $O\left(MN\left(\log MN\right)\right)$, and in practical experiments, the GPFIM model operates 2 times faster.

The GSFM model provides high-quality, predictable, and efficient fractal texture generation for jewelry. The model automatically adapts to various production technologies and is ready for use in professional design processes.

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