PROPERTIES OF THE SOLUTION TO A WAVE EQUATION GIVEN ON A CIRCLE AND METHODS FOR REGULARIZING INVERSE PROBLEMS

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Abstract. *This paper investigates the analytical properties of solutions to the wave equation defined on a bounded trajectory, focusing specifically on how boundary observations can be utilized to reconstruct internal medium parameters. We examine the forward problem's well-posedness and contrast it with the inherent ill-posedness of the corresponding inverse problem. Furthermore, we discuss advanced mathematical frameworks designed to stabilize parameter recovery under the presence of measurement noise, emphasizing boundary control methodologies and regularized minimization strategies.*

Keywords: *wave equation, inverse problems, boundary control method, regularization, boundary observations, Neumann-to-Dirichlet map, Hadamard stability, convergence rates.*

Introduction. The study of hyperbolic partial differential equations on bounded domains or compact manifolds forms the mathematical foundation for various physical phenomena, including acoustic propagation, seismic imaging, and non-destructive testing. When the spatial domain is bounded—such as a circle or a compact manifold with a boundary—the propagation of waves carries signature geometric and structural information back to the boundary. The forward problem involves determining the wave field inside the domain given known initial states and boundary conditions. Conversely, the inverse problem aims to reconstruct unknown internal coefficients—such as a variable wave speed—solely from observations collected at the boundary.

Because inverse problems inherently violate Hadamard's stability criterion, direct inversion methods typically amplify data perturbations and measurement noise. To overcome these challenges, specialized mathematical frameworks known as regularization strategies must be deployed to guarantee the continuous dependency of the reconstructed parameters on the collected boundary data.

Mathematical Framework and Direct Problem Properties

Consider a wave equation defined on a compact domain or a closed circular trajectory. In an isotropic medium, the propagation of disturbances can be modeled by the classical hyperbolic equation:

$$\partial^2 u(t, x)/\partial t^2 - c(x)^2 \cdot \Delta u(t, x) = 0$$

where c(x) represents the spatially varying wave speed. When analyzing inverse boundary value problems within this geometry, a common operational framework is the hyperbolic Neumann-to-Dirichlet (NtD) map. This operator physically maps an applied boundary source (Neumann data) to the observed wave field restricted to the accessible boundary (Dirichlet data).

The direct problem under this framework is well-posed in the sense of Jacques Hadamard. Given a sufficiently smooth velocity function c(x) within an admissible model space, there exists a unique, stable solution u(t,x) that continuously depends on the initial and boundary conditions. The finite propagation speed of the wave ensures that a disturbance initiated at a specific boundary region requires a deterministic travel time to reach internal points and echo back, establishing a direct link between time-dependent boundary measurements and internal spatial depths.

Ill-Posedness of the Inverse Problem

The objective of the inverse problem is to invert the forward mapping $A: c \mapsto \Lambda$, where Λ represents the NtD map corresponding to the internal wave speed c(x). While the forward operator is continuous, its inverse A^{-1} lacks continuity across the entire data space.

In practical scenarios, exact boundary operators are unreachable due to unavoidable measurement noise and discretization errors. The actual observed data can be modeled as $\tilde{\Lambda} = \Lambda + E$, where E represents a linear operator bounding the measurement error such that $||E|| \leq \varepsilon$. Because $\tilde{\Lambda}$ typically falls outside the range of the true forward operator A, a direct application of an unregularized inverse mapping fails completely, as small data perturbations can produce arbitrarily large errors in the reconstructed velocity profile.

4. Methods for Regularizing Inverse Problems

To restore stability to the inversion process, a family of bounded, continuous mapping operators R_α must be constructed to approximate A^{-1}. A key methodology utilized to solve such non-linear hyperbolic inverse problems is the boundary control (BC) method and its iterative variations.

The boundary control variant relies heavily on Blagovestchenskii-type identities, which allow the computation of inner products of wave fields inside the hidden medium solely by utilizing data accessible at the boundary. By setting up a regularized minimization functional, one can identify specific boundary source sequences that force the internal wave field to approximate indicator functions of the domain of influence. Specifically, a regularized optimization problem can be formulated on the boundary as:

$$min\_h \;\; (2\langle Ph, \hat{P}\Phi\rangle \; + \; \langle Ph, KPh\rangle \; + \; \alpha\langle h, h\rangle)$$

where K represents a boundary operator synthesized from the time-reversal map and the NtD map, $\alpha > 0$ is the regularization parameter, and h is the boundary source.

As established in contemporary regularization theory, carefully coupling the regularization parameter α to the known noise level ε ensures that the reconstructed profile converges precisely to the true wave speed c(x) as the noise approaches zero. This type of direct regularization method avoids local optimization traps and guarantees robust convergence rates under specified Hölder-type bounds.

Conclusion. Solving the inverse problem for the wave equation on circular or compact domains requires a deep synergy between partial differential equations and regularization theory. While forward wave propagation is inherently stable, recovering internal characteristics from boundary observations is profoundly ill-posed. Utilizing boundary control techniques combined with regularized optimization frameworks provides a structurally sound and stable framework capable of reconstructing internal parameters even from highly perturbed data sets.

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