CLASSIFICATION AND LIE STRUCTURES OF GIVEN SOLVABLE LEIBNIZ ALGEBRAS WITH NIL RADICALS

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ABSTRACT: *This work is devoted to the classification of solvable Leibniz algebras with an abelian nilradical. We consider (***k***–1)-dimensional extension of* **k***-dimensional abelian algebras and classify all (2***k***–1)-dimensional solvable Leibniz algebras with an abelian nilradical of dimension* **k***.*

KEYWORDS: *Derivation; Leibniz algebra; nilpotency; nilradical; solvability*

1. Introduction

Leibniz algebras, a "noncommutative version" of Lie algebras, were first introduced in the mid- 1960s by Blokh [5] under the name of "*D*-algebras." They appeared again in the 1990s after Loday's work [12], where he introduced calling them Leibniz algebras.

According to the structural theory of Lie algebras, a finite-dimensional Lie algebra can be writ- ten as a semidirect sum of its semisimple subalgebra and its solvable radical (Levi's theorem). The semisimple part is a direct sum of simple Lie algebras which were completely classified in the fifties of the last century (see [9]). In the case of Leibniz algebras, there is also an analog to Levi's theorem [3]. Namely, the decomposition of a Leibniz algebra into the semidirect sum of its solvable radical and a semisimple Lie algebra can be obtained. As above, the semisimple part can be composed by simple Lie algebras and the main issue in the classification problem of finite- dimensional complex Leibniz algebras is to study the solvable part. Therefore, the classification of solvable Leibniz algebras is important to construct finite-dimensional Leibniz algebras.

Owing to a result of [13], an approach to the study of solvable Lie algebras through the use of the nilradical was developed in [2, 14, 17], etc. In particular, in [14] solvable Lie algebras with abelian nilradicals are investigated.

The analog of Mubarakzjanov's result has been applied in the Leibniz algebra case in [7], showing the importance of consideration of the nilradical in the case of Leibniz algebras as well. The papers [6, 7, 11, 15, 16] also are devoted to the study of solvable Leibniz algebras by consid- ering their nilradicals.

It should be noted that any solvable Leibniz algebra L with nilradical N can be written as a direct sum of vector spaces $L = N$ 8 Q, where Q is the complementary vector space to the nilradical. In [1, 4], solvable Leibniz algebras with an abelian nilradical are investigated. It was proven that the maximal dimension of a solvable Leibniz algebra with k-dimensional abelian nilradical is $2k$. Additionally, in [1] this maximal case was classified and some results regarding of the classification

with one-dimensional extension were presented. In this paper, we give the classification of solvable Leibniz algebras with abelian nilradical and $(k — 1)$-dimensional extension.

It should be noted that a solvable Leibniz algebra L with condition $\dim Q = \dim(N/N^2)$ can be classified using the classification of solvable Leibniz algebras with a k-dimensional abelian nil- radical and a k-dimensional complementary vector space.

The natural next step is the classification of solvable Leibniz algebra with condition $\dim Q = \dim(N/N^2) — 1$. In order to perform this classification, the classification of solvable Leibniz alge- bras with a k-dimensional abelian nilradical and a $(k — 1)$-dimensional complementary vector

space should first be obtained. In the case $k = 2$ and $k = 3$ we have three- and five-dimensional solvable Leibniz algebras, which were classified in [8] and [10], respectively. In this paper, we consider the case for any k, i.e., we classify all $(2k — 1)$-dimensional solvable Leibniz algebras with k-dimensional abelian nilradical.

Throughout this paper all algebras (vector spaces) considered are finite-dimensional and over the field of complex numbers. Also, in the tables of multiplications of algebras, we give nontrivial products only.

2. **Preliminaries**

This section is devoted to recalling some basic notions and concepts used throughout the paper.

Definition 2.1. A C-vector space with a bilinear bracket $(L, [\cdot, \cdot])$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] — [[x, z], y],$$

holds.

Here, we adopt the right Leibniz identity; since the bracket is not skew-symmetric, there exists the version corresponding to the left Leibniz identity

$$[[x, y], z] = [x, [y, z]] — [y, [x, z]].$$

Note that the notions of ideal and subalgebra are defined by the usual way. The sets $Ann_r(L) := \{x \in L : [L, x] = 0\}$ and $Ann_l(L) := \{x \in L : [x, L] = 0\}$ are called the *right* and *left annihilators* of L, respectively. It is observed that $Ann_r(L)$ is a two-sided ideal of L, and for any $x, y \in L$ the elements $[x, x]$ and $[x, y] + [y, x]$ are always in $Ann_r(L)$.

The set $C(L) := \{x \in L : [x, L] = [L, x] = 0\}$ is called the *center* of L.

For a given Leibniz algebra $(L, [\cdot, \cdot])$ the sequences of two-sided ideals is defined recursively as follows:

$$L^1 := L, L^{k+1} := [L^k, L], k \geq 1, \qquad L^{[1]} := L, L^{[s+1]} := [L^{[s]}, L^{[s]}], s \geq 1.$$

These are said to be the *lower central* and the *derived* series of L, respectively.

Definition 2.2. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists

$n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$).

Definition 2.3. An ideal of a Leibniz algebra is called nilpotent if it is nilpotent as a subalgebra.

It is well-known that the sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

Definition 2.4. The maximal nilpotent ideal N of a Leibniz algebra L is said to be the nilradical of the algebra.

Definition 2.5. A linear map $d : L \to L$ of a Leibniz algebra $(L, [\cdot, \cdot])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$d([x, y]) = d(x), y + x, d(y)$.

The set of all derivations of L is denoted by Der(L) and it is a Lie algebra with respect to the commutator.

For a given element x of a Leibniz algebra L, the right multiplication operator $R_x : L \to L$, defined by $R_x(y) = [y, x], y \in L$ is a derivation. In fact, Leibniz algebras are characterized by this property regarding right multiplication operators. (Recall that left Leibniz algebras are character- ized by the same property with left multiplication operators.) As in the Lie case, such kind of der- ivations are said to be inner derivations.

Definition 2.6. Let $d_1, d_2, ..., d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, ..., d_n$

are said to be linearly nil-independent if for $a_1, a_2, ..., a_n \in$ C and a natural number k,

$(a_1d_1 + a_2d_2 + \cdots + a_nd_n)^k = 0 implies\ a_1 = a_2 = \cdots = a_n = 0.$

Note that in the above definition the power is understood with respect to composition.

Let L be a solvable Leibniz algebra. Then it can be written in the form $L = N$ 8 Q, where N is the nilradical and Q is the complementary vector subspace. The following is a result from [7] on the dimension of Q which we make use of in the paper.

Theorem 2.7. *Let L be a solvable Leibniz algebra and N be its nilradical. Then the dimension of Q is not greater than the maximal number of nil-independent derivations of N.*

3. Main result

We denote by a_k a k-dimensional abelian algebra and by $R(a_k, s)$ the class of solvable Leibniz algebras with k-dimensional abelian nilradical N and s-dimensional complementary vector space Q.

As above, it has been proven that $s \leq k$ for any algebra from the class $R(a_k, s)$, and in [1] the classification of such algebras of $R(a_k, k)$ is given. It is proven that an arbitrary algebra from the family $R(a_k, k)$ can be decomposed into a direct sum of copies of two-dimensional non-trivial solvable Leibniz algebras.

It is proven that an arbitrary algebra L from the family $R(a_k, k)$ is

$L = l_2$ 8 l_2 8 $\cdots$ 8 l_2 8 r_2 8 r_2 8 $\cdots$ 8 r_2, where $l_2 : [e, x] = e$ and r_2 : $[e, x] = —[x, e] = e$.

Let L be a Leibniz algebra from the class $R(a_k, k — 1)$. Take a basis $\{e_1, e_2, ..., e_k, x_1, x_2, ..., x_{k—1}\}$

of L such that $\{e_1, e_2, ..., e_k\}$ is a basis of nilradical N and $\{x_1, x_2, ..., x_{k—1}\}$ is a basis of the com- plementary vector space Q. It is known that the right multiplication operators R_{x1}, R_{x2}, ..., $R_{xk—1}$: $N \to N$ are nil-independent derivations [7] and there exists a basis of N, for an easier notation, suppose again $\{e_1, e_2, ..., e_k\}$, such that operators R_{x1}, R_{x2}, ..., $R_{xk—1}$ simultaneously have the Jordan normal form.

Because of this, observe that:

$$\begin{cases} [e_i, x_j] = a_{i,j}e_i + b_{i,j}e_{i+1}, 1 \leq i, j \leq k — 1, \\ [e_k, x_j] = a_{k,j}e_k \qquad 1 \leq j \leq k — 1, \end{cases}$$

where $a_{i,j}$ are eigenvalues of the operator R_{xj} and $b_{i,j} \in \{0, 1\}$.

Since $R_{x_1}, R_{x_2}, \dots, R_{x_{k-1}}$ are nil-independent we have that

$$rank \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k-1} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k-1} \\ \dots & \dots & \dots & \dots \\ a_{k-1,1} & a_{k-1,2} & \dots & a_{k-1,k-1} \\ a_{k,1} & a_{k,2} & \dots & a_{k,k-1} \end{pmatrix} = k-1.$$

Thus, there exists a minor of order $k-1$ which has a non-zero determinant, i.e., there exists t such that $det(M^{k-1}_{i_1,i_2,\dots,i_{k-1}}) \neq 0$. Making the change of basis

$$\begin{cases} e'_i = e_i, & 1 \le i \le t-1, \\ e'_i = e_{i+1}, & t \le i \le k-1, \\ e'_k = e_t, \end{cases}$$

we get that

$$\begin{cases} [e_i, x_j] = a_{i,j}e_i + b_{i,j}e_{i+1}, & 1 \le i, j \le k-1, \ i \neq t-1, \\ [e_{t-1}, x_j] = a_{t-1,j}e_{t-1} + b_{t-1,j}e_k, & 1 \le j \le k-1, \\ [e_k, x_j] = a_{k,j}e_k + b_{k,j}e_t, & 1 \le j \le k-1. \end{cases}$$

It should be noted that operators $R_{x_1}, R_{x_2}, \dots, R_{x_{k-1}}$ can be considered linearly nil-independent operators on the quotient vector space $a_k/\langle e_k\rangle$. Since $\dim(a_k/\langle e_k\rangle) = k-1$ from the result of [1] we obtain that

$$a_{i,i} = 1, \quad 1 \le i \le k-1, \qquad a_{i,j} = 0, \quad 1 \le i, j \le k-1, \quad i \neq j,$$

$$b_{i,j} = 0, \quad 1 \le i \le k, \quad 1 \le j \le k-1, \quad i \neq t-1.$$

Let us introduce the following notation:

$$[x_i, e_j] = \sum_{p=1}^{k} c^p_{i,j} e_p, \quad 1 \le i \le k-1, \quad 1 \le j \le k, \quad c^p_{i,j} \in \mathbb{C}.$$

Using the similar algorithms of the proof of Theorem 3.2 in [1], which was given the classification of solvable Leibniz algebras $R(a_k, k)$, from Leibniz identities and basis changes we obtain that

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$i\ i,i$ c
$p\ i,j = 0,\ d\ 1 \le i,j,p \le k-1, \quad i \ne j \ne p,$
$\in \{0, -1\}, \qquad 1 \le i \le k-1,$
$= 0, \qquad 1 \le i,j,p \le k-1.$

Therefore, the multiplication of the $(2k-1)$-dimensional solvable Leibniz algebras with k- dimensional abelian nilradical N has the following form

$$R(a_k, k-1) = \begin{cases} [e_i, x_i] = e_i + b_{i,i}e_k, & 1 \le i \le k-1, \\ [e_i, x_j] = b_{i,j}e_k, & 1 \le i \le k, \quad 1 \le j \le k-1, \quad i \ne j, \\ [x_i, e_i] = a_i e_i + c_{i,i} e_k, & 1 \le i \le k-1, \\ [x_i, e_j] = c_{i,j}e_k, & 1 \le i \le k-1, \quad 1 \le j \le k-1, \quad i \ne j, \\ [x_i, e_k] = \sum_{j=1} \ i_{,j}e_i, & 1 \le i \le k-1, \\ [x_i, x_j] = d_{i,j}e_k, & 1 \le i,j \le k-1, \end{cases} \quad (3.1)$$

$[x_i, x_j] = d_{i,j}e_k, 1 \le i,j \le k-1,$

where $a_i \in \{0, -1\}$.

First we investigate the case of $a_i = 0$ for all $1 \le i \le k-1$.

Theorem 3.1. *Let L be a Leibniz algebra from the class* $R(a_k, k-1)$ *and let* $a_i = 0$ *for* $1 \le i \le k-1$. *Then L is isomorphic to one of the following algebras:*

$$L_1(b_i): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_i] = b_i e_k, & 1 \le i \le k-1, \end{cases} \qquad L_2(b_i): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_i] = b_i e_k, & 1 \le i \le k-1, \\ [x_i, e_k] = -b_i e_k, & 1 \le i \le k-1, \end{cases}$$

$$L_3(b_i): \begin{cases} [e_1, x_1] = e_1 + b_1 e_k, \\ [e_i, x_i] = e_i, & 2 \le i \le k-1, \\ [e_1, x_i] = b_i e_k, & 2 \le i \le k-1, \\ [e_k, x_1] = e_k, \end{cases} \qquad L_4(\ ): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_1] = e_k, \\ [x_1, e_k] = -e_k, \\ [x_i, e_k] = \ _i e_1, & 2 \le i \le k-1, \end{cases}$$

$$L_5(d_{i,j}): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [x_i, x_j] = d_{i,j} e_k, & 1 \le i,j \le k-1. \end{cases}$$

Proof. Let $a_i = 0$ for $1 \le i \le k-1$, then the multiplication (3.1) has the form

$$[e_i, x_i] = e_i + b_{i,i}e_k, \quad 1 \leq i \leq k-1,$$
$$[e_i, x_j] = b_{i,j}e_k, \quad 1 \leq i \leq k, \quad 1 \leq j \leq k-1, \ i \neq j,$$
$$[x_i, e_j] = c_j e, \quad 1 \leq i \leq k-1, \ 1 \leq j \leq k-1,$$
$$[x_i, e_k] = \sum_{j=1} \ {}_{i,j}e_j, \quad 1 \leq i \leq k-1,$$
$$[x_i, x_j] = d_{i,j}e_k, \quad 1 \leq i, j \leq k-1.$$

Case 1. Let there exist $i_0 \in \{1, 2, ..., k\}$, such that $b_{k, i0} \notin \{0, 1\}$. Without loss of generality, we may assume $i_0 = 1$. Making the change of basis

$$e' = e_1 - \frac{b_{1,1}}{} e_k, \qquad e'_i = e_k - \frac{b_{i,1}}{} e, \ 2 \leq i \leq k-1,$$

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