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ON THE POINT SPECTRUM OF THE FINITE RANK FREDHOLM INTEGRAL OPERATOR

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Annotation. *In the present note it is considered the Fredholm integral operator T with, rank $n, n \in N$ in the Hilbert space $L_2[-\pi;\pi]$. Firstly, it is mentioned that the number 0 is an eigenvalue of the Fredholm integral operator $T = T_1 + T_2 + \dots + T_n$ with infinite multiplicity. The Fredholm determinant corresponding to this operator is constructed and its point spectrum is described.*

Keys words: *integral operator, kernel, Fredholm operator, parameter function, rank of operator.*

In mathematics, Fredholm operators are certain operators that arise in the Fredholm theory of integral equations. In the present note we consider the finite rank Fredholm integral operator and describe its spectrum.

In the Hilbert space $L_2[-\pi;\pi]$ we consider the operator of the form

$$(T_i f)(x) = v_i(x) \int_{-\pi}^{\pi} v_i(t) f(t) dt$$

for $i = 1,2,\dots,n$. Here the functions $v_i(\cdot)$, $i = 1,2,\dots,n$ are real-valued continuous and linearly independent functions defined on $[-\pi;\pi]$.

We note that the scalar product of the two elements f and g from $L_2[-\pi;\pi]$ is defined by

$$(f,g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt.$$

Analogously, the norm of the element $f \in L_2[-\pi;\pi]$ is defined by

$$\|f\| = \left(\int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{\frac{1}{2}}$$

Using these formulas and corresponding definitions one can show that the operator T acting in the Hilbert space $L_2[-\pi;\pi]$ as

$$T = T_1 + T_2 + \dots + T_n$$

is linear, bounded and self-adjoint.

By the construction the equality

$$(Tf)(x)=\sum_{i=1}^{n} v_i(x)\int_{-\pi}^{\pi} v_i(t)\,f(t)dt$$

holds.

Recall that the number $\lambda = 0$ is an eigenvalue of the operator T with infinite multiplicity; an infinite set of (orthogonal) eigenfunctions f_m $(m = 1,2,\dots)$ characterized by

$$0=(f_m, v_i)=\int_{-\pi}^{\pi} v_i(x)f_m(x)dx\,,\qquad i=1,2,\dots,n.$$

There are n non-zero eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$ which are zeros of the function

$$\Delta(\lambda) \coloneqq \det\left(\lambda\delta_{ij}-\left(v_i,v_j\right)\right)_{i,j=1}^{n},$$

where

$$\delta_{i,j} \coloneqq \begin{cases} 1, & if \quad i=j, \\ 0, & if \quad i\neq j. \end{cases}$$

Usually the function $\Delta(\cdot)$ is called a Fredholm determinant associated with the operator T.

For the discrete spectrum of the operator T we have the following equality

$$\sigma_{\text{disc}}(T)=\{\lambda\neq 0\colon\ \Delta(\lambda)=0\}.$$

Also for the essential spectrum of T we have

$$\sigma_{\text{ess}}(T)=\{0\}.$$

For the sake of convenience in our further research, we require the condition

$\left(v_i,v_j\right)=0,\ \ i\neq j,\ \ i,j=1,2,\dots,n$ (1)

to be fulfilled. Then the function $\Delta(\cdot)$ can be rewritten as a product of the form

$$\Delta(\lambda)=(\lambda-\|v_1\|^2)(\lambda-\|v_2\|^2)\dots\dots\dots(\lambda-\|v_n\|^2).$$

Therefore

$$\sigma_{\text{disc}}(T)=\{\,\|v_1\|^2,\|v_2\|^2,\dots,\|v_n\|^2\},$$

$$\sigma(T)=\sigma_{\text{pp}}(T)=\{0,\|v_1\|^2,\|v_2\|^2,\dots,\|v_n\|^2\}.$$

Simple calculations show that

$$\sigma(T_i)=\sigma_{\text{pp}}(T_i)=\{0,\|v_i\|^2\},\quad i=1,2,\dots,n.$$

The main result of the present note is the following theorem.

Theorem. If the condition (1) is fulfilled, then the equality

$$\sigma(T)=\sigma(T_1)\cup\sigma(T_2)\cup\dots\cup\sigma(T_n)$$

holds for the spectrum of T.

We notice that such type Fredholm integral operators can be considered as perturbation operator in the Friedrichs model [1-4].

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