THE EXTENSIVE FORM OF A GAME

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Annotation: *This article explores the two fundamental models of game theory — the strategic form and the extensive form. While the strategic form provides a compact mathematical representation of a game, it fails to capture crucial aspects of real-world interactions such as bluffing, signaling, and sequential decision-making. The extensive form, on the other hand, illustrates the game through a game tree, incorporating chance moves, information sets, and the temporal structure of decisions, thus offering a richer and more realistic model of strategic behavior.*

Keywords: *Game theory, strategic form, extensive form, game tree, information set, bluffing, signaling, decision-making*

The strategic form of a game is a compact way of describing the mathematical aspects of a game. In addition, it allows a straightforward method of analysis, at least in principle. However, the flavor of many games is lost in such a simple model. Another mathematical model of a game, called the extensive form, is built on the basic notions of position and move, concepts not apparent in the strategic form of a game. In the extensive form, we may speak of other characteristic notions of games such as bluffing, signaling, sandbagging, and so on. Three new concepts make their appearance in the extensive form of a game: the game tree, chance moves, and information sets.

1 The Game Tree. The extensive form of a game is modelled using a directed graph. A directed graph is a pair (T, F) where T is a nonempty set of vertices and F is a function that gives for each $x \in T$ a subset, F(x) of T called the followers of x. When a directed graph is used to represent a game, the vertices represent positions of the game. The followers, F(x), of a position, x, are those positions that can be reached from x in one move.

A path from a vertex t_0 to a vertex t_1 is a sequence, $\mathbf{x_0}, \mathbf{x_1}, \ldots, \mathbf{x_n}$, of vertices such that $x_0 = t_0$, $x_n = t_1$ and x_i is a follower of x_{i-1} for i = 1, . . . , n. For the extensive form of a game, we deal with a particular type of directed graph called a tree.

Definition. A tree is a directed graph, (T, F) in which there is a special vertex, t0, called the root or the initial vertex, such that for every other vertex t ∈ T, there is a unique path beginning at t0 and ending at t.

The existence and uniqueness of the path implies that a tree is connected, has a unique initial vertex, and has no circuits or loops.

In the extensive form of a game, play starts at the initial vertex and continues along one of the paths eventually ending in one of the terminal vertices. At terminal vertices, the rules of the game specify the payoff. For n-person games, this would be an n-tuple of payoffs. Since we are dealing with two-person zero-sum games, we may take this payoff to be the amount Player I wins from Player II. For the nonterminal vertices there are three possibilities. Some nonterminal vertices are assigned to Player I who is to choose the move at that position. Others are assigned to Player II. However, some vertices may be singled out as positions from which a chance move is made.

Chance Moves. Many games involve chance moves. Examples include the rolling of dice in board games like monopoly or backgammon or gambling games such as craps, the dealing of cards as in bridge or poker, the spinning of the wheel of fortune, or the drawing of balls out of a cage in lotto. In these games, chance moves play an important role. Even in chess, there is generally a chance move to determine which player gets the white pieces (and therefore the first move which is presumed to be an advantage). It is assumed that the players are aware of the probabilities of the various outcomes resulting from a chance move.

Information. Another important aspect we must consider in studying the extensive form of games is the amount of information available to the players about past moves of the game. In poker for example, the first move is the chance move of shuffling and dealing the cards, each player is aware of certain aspects of the outcome of this move (the cards he received) but he is not informed of the complete outcome (the cards received by the other players). This leads to the possibility of "bluffing."

2 Basic Endgame in Poker. One of the simplest and most useful mathematical models of a situation that occurs in poker is called the "classical betting situation" by Friedman (1971) and "basic endgame" by Cutler (1976). These papers provide explicit situations in the game of stud poker and of lowball stud for which the model gives a very accurate description. This model is also found in the exercises of the book of Ferguson (1967). Since this is a model of a situation that occasionally arises in the last round of betting when there are two players left, we adopt the terminology of Cutler and call it Basic Endgame in poker. This will also emphasize what we feel is an important feature of the game of poker, that like chess, go, backgammon and other games, there is a distinctive phase of the game that occurs at the close, where special strategies and tactics that are analytically tractable become important.

Basic Endgame is played as follows. Both players put 1 dollar, called the ante, in the center of the table. The money in the center of the table, so far two dollars, is called the pot. Then Player I is dealt a card from a deck. It is a winning card with probability 1/4 and a losing card with probability 3/4. Player I sees this card but keeps it hidden from Player II. (Player II does not get a card.) Player I then checks or bets. If he checks, then his card is inspected; if he has a winning card he wins the pot and hence wins the 1 dollar ante from II, and otherwise he loses the 1 dollar ante to II. If I bets, he puts 2 dollars more into the pot. Then Player II – not knowing what card Player I has – must fold or call. If she folds, she loses the 1 dollar ante to I no matter what card I has. If II calls, she adds 2 dollars to the pot. Then Player I's card is exposed and I wins 3 dollars (the ante plus the bet) from II if he has a winning card, and loses 3 dollars to II otherwise.

Let us draw the tree for this game. There are at most three moves in this game: (1) the chance move that chooses a card for I, (2) I's move in which he checks or bets, and (3) II's move in which she folds or calls. To each vertex of the game tree, we attach a label indicating which player is to move from that position. Chance moves we generally refer to as moves by nature and use the label N. See Figure 1.

Each edge is labelled to identify the move. (The arrows are omitted for the sake of clarity. Moves are assumed to proceed down the page.) Also, the moves leading from a vertex at which nature moves are labelled with the probabilities with which they occur. At each terminal vertex, we write the numerical value of I's winnings (II' s losses).

There is only one feature lacking from the above figure. From the tree we should be able to reconstruct all the essential rules of the game. That is not the case with the tree of Figure 1 since we have not indicated that at the time II makes her decision she does not know which card I has received. That is, when it is II's turn to move, she does not know at

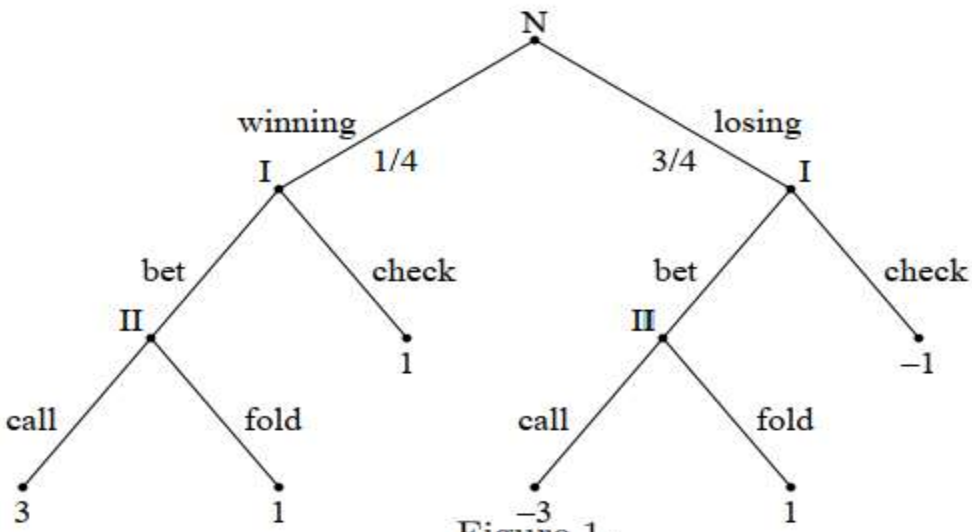


Figure 1.

which of her two possible positions she is. We indicate this on the diagram by encircling the two positions in a closed curve, and we say that these two vertices constitute an information set. The two vertices at which I is to move constitute two separate information sets since he is told the outcome of the chance move. To be complete, this must also be indicated on the diagram by drawing small circles about these vertices. We may delete one of the labels indicating II's vertices since they belong to the same information set. It is really the information set that must be labeled. The completed game tree becomes

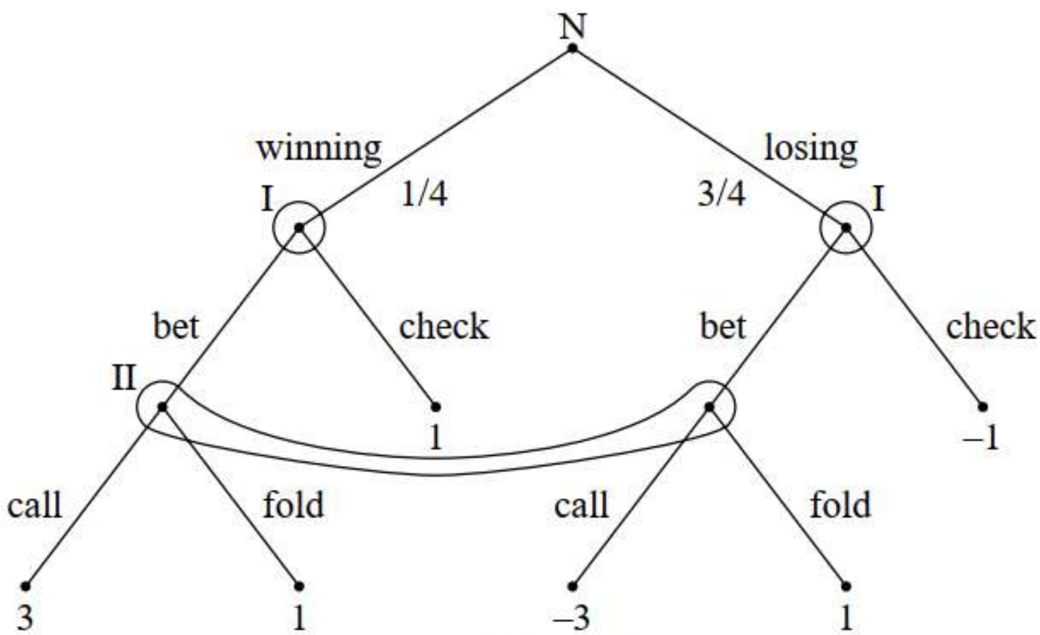


Figure 2.

The diagram now contains all the essential rules of the game

3 The Kuhn Tree. The game tree with all the payoffs, information sets, and labels for the edges and vertices included is known as the Kuhn Tree. We now give the formal definition of a Kuhn tree.

Not every set of vertices can form an information set. In order for a player not to be aware of which vertex of a given information set the game has come to, each vertex in that information set must have the same number of edges leaving it. Furthermore, it is important that the edges from each vertex of an information set have the same set of labels. The player moving from such an information set really chooses a label. It is presumed that a player makes just one choice from each information set.

Definition. A finite two-person zero-sum game in extensive form is given by

1) a finite tree with vertices T,

2) a payoff function that assigns a real number to each terminal vertex,

3) a set T_0 of non-terminal vertices (representing positions at which chance moves occur) and for each t ∈ T0, a probability distribution on the edges leading from t,

4) a partition of the rest of the vertices (not terminal and not in T0) into two groups of information sets $T_{11}, T_{12}, \ldots, T_{1k_1}$ (for Player I) and $T_{21}, T_{22}, \ldots, T_{2k_2}$ (for Player II), and

5) for each information set T_{jk} for player j, a set of labels L_{jk}, and for each t ∈ T_{jk} , a one-to-one mapping of L_{jk} onto the set of edges leading from t.

The information structure in a game in extensive form can be quite complex. It may involve lack of knowledge of the other player's moves or of some of the chance moves. It may indicate a lack of knowledge of how many moves have already been made in the game (as is the case With Player II in Figure 3).

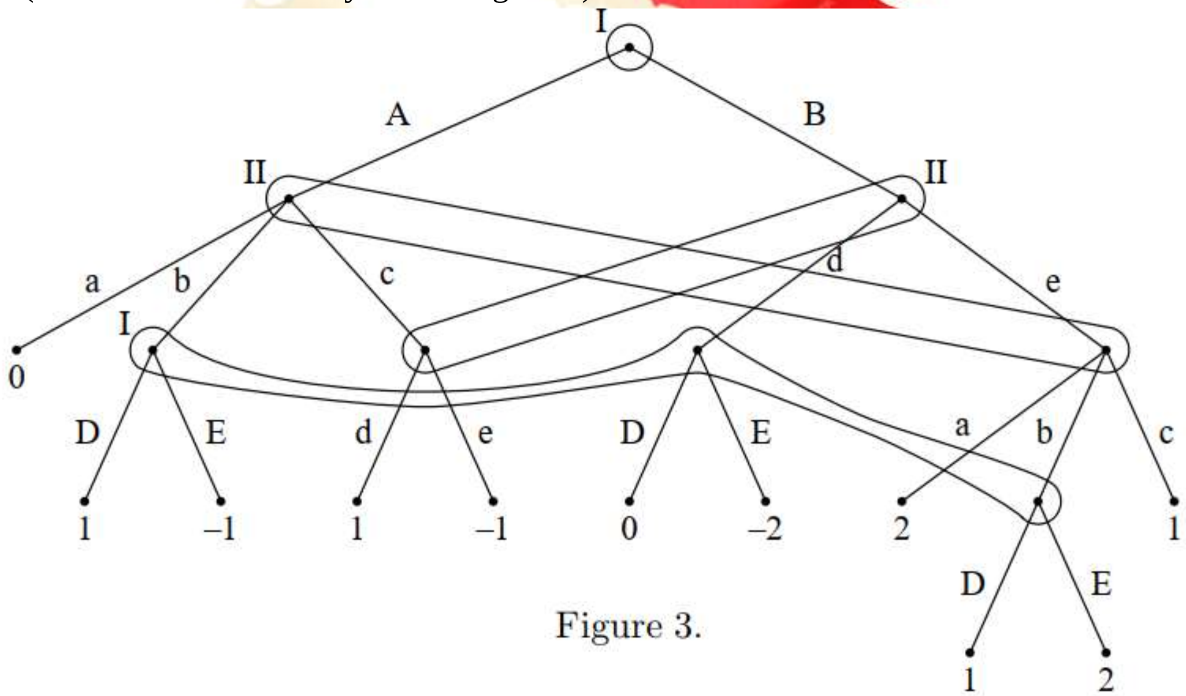


Figure 3.

It may describe situations in which one player has forgotten a move he has made earlier (as is the case With Player I in Figures 3 or 4). In fact, one way to try to model the game of bridge as a two-person zero-sum game involves the use of this idea. In bridge, there are four individuals forming two teams or partnerships of two players each. The interests of the members of a partnership are identical, so it makes sense to describe this as a two-person game. But the members of one partnership make bids alternately based on cards that one member knows and the other does not. This may be described as a single player who alternately remembers and forgets the outcomes of some of the previous random moves. Games in which players remember all past information they once knew and all past moves they made are called games of perfect recall.

A kind of degenerate situation exists when an information set contains two vertices which are joined by a path, as is the case with I's information set in Figure 5.

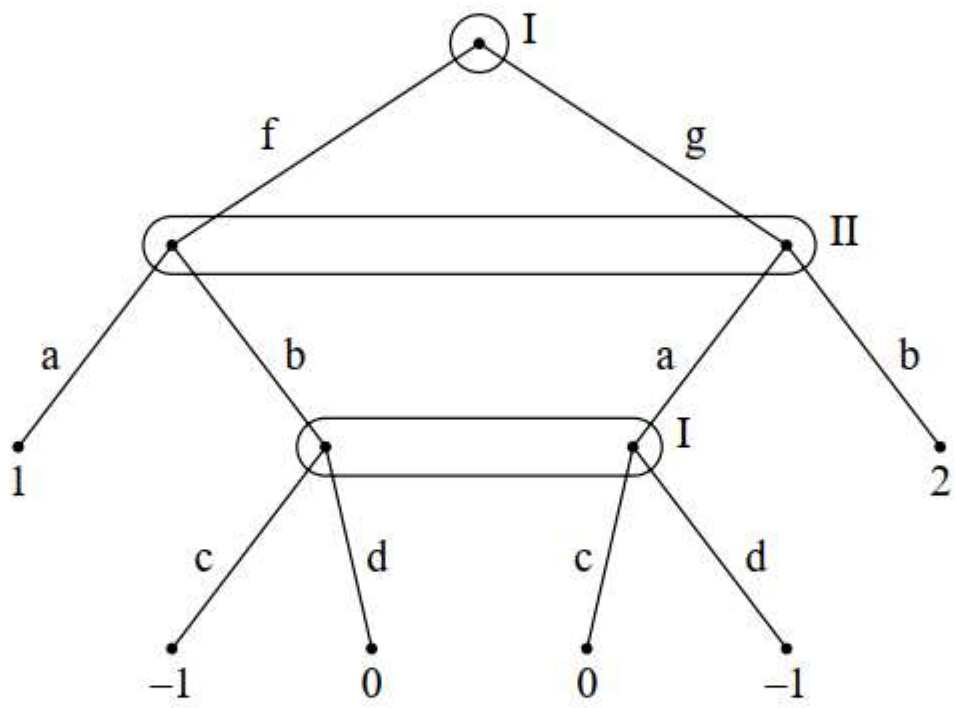


Figure 4.

We take it as a convention that a player makes one choice from each information set during a game. That choice is used no matter how many times the information set is reached. In Figure 5, if I chooses option a there is no problem. If I chooses option b, then in the lower of I's two vertices the a is superfluous, and the tree is really equivalent to Figure 6. Instead of using the above convention, we may if we like assume in the definition of a game in extensive form that no information set contains two vertices joined by a path.

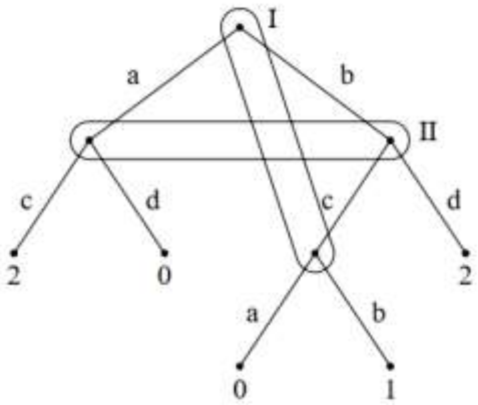


Figure 5.

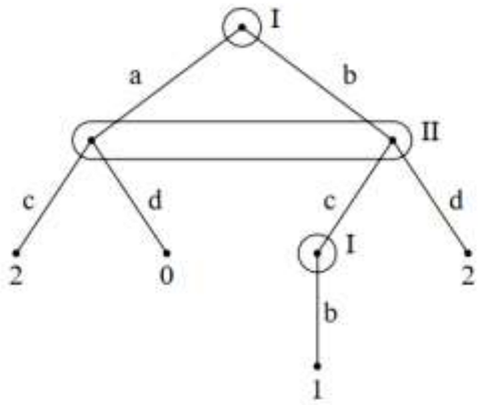
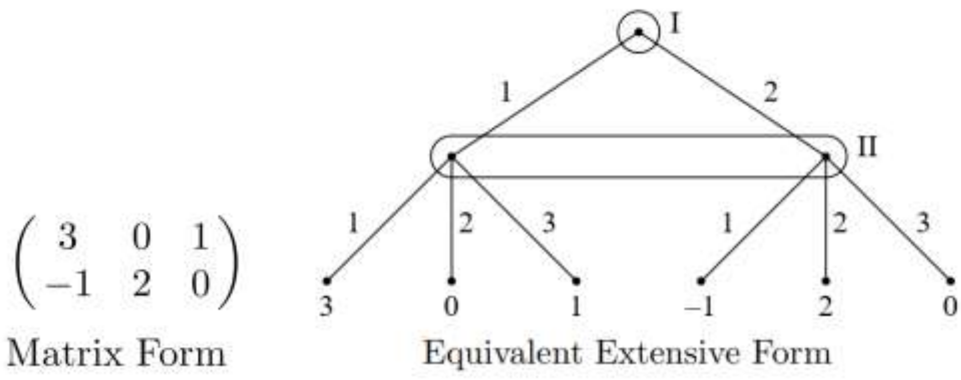


Figure 6.

Games in which both players know the rules of the game, that is, in which both players know the Kuhn tree, are called games of complete information. Games in which one or both of the players do not know some of the payoffs, or some of the probabilities of chance moves, or some of the information sets, or even whole branches of the tree, are called games with incomplete information, or pseudogames. We assume in the following that we are dealing with games of complete information.

4 The Representation of a Strategic Form Game in Extensive Form. The notion of a game in strategic form is quite simple. It is described by a triplet (X, Y, L) as in Section 1. The extensive form of a game on the other hand is quite complex. It is described by the game tree with each non-terminal vertex labeled as a chance move or as a move of one of the players, with all information sets specified, with probability distributions given for all chance moves, and with a payoff attached to each terminal vertex. It would seem that the theory of games in extensive is much more comprehensive than the theory of games in strategic form. However, by taking a game in extensive form and considering only the strategies and average payoffs, we may reduce the game to strategic form.

First, let us check that a game in strategic form can be put into extensive form. In the strategic form of a game, the players are considered to make their choices simultaneously, while in the extensive form of a game simultaneous moves are not allowed. However, simultaneous moves may be made sequentially as follows. We let one of the players, say Player I, move first, and then let player II move without knowing the outcome of I's move. This lack of knowledge may be described by the use of an appropriate information set. The example below illustrates this.



Player I has 2 pure strategies and Player II has 3. We pretend that Player I moves first by choosing row 1 or row 2. Then Player II moves, not knowing the choice of Player I. This is indicated by the information set for Player II. Then Player II moves by choosing column 1, 2 or 3, and the appropriate payoff is made.

5 Reduction of a Game in Extensive Form to Strategic Form. To go in the reverse direction, from the extensive form of a game to the strategic form, requires the consideration of pure strategies and the usual convention regarding random payoffs.

Pure strategies. Given a game in extensive form, we first find X and Y , the sets of pure strategies of the players to be used in the strategic form. A pure strategy for Player I is a rule that tells him exactly what move to make in each of his information sets. Let $T_{11}, \ldots, T_{1k_1}$ be the information sets for Player I and let $L_{11}, \ldots, L_{1k_1}$ be the corresponding sets of labels. A pure strategy for I is a k_1-tuple $x=(x_1,\ldots,x_{k_1})$ where for each i, x_i is one of the elements of L_{1i}. If there are m_i elements in L_{1i}, the number of such k_l-tuples and hence the number of I s pure strategies is the product $m_1, m_2, \ldots, m_k$. The set of all such strategies is X. Similarly, if $T_{21}, \ldots, T_{2k}$ represent II's information sets and $L_{21}, \ldots, L_{2k_2}$ the corresponding sets of labels, a pure strategy for II is a k_2-tuple, $y = (y_1, \ldots, y_k)$ where $y \in L_{2j}$ for each j. Player II has $n_1, n_2, \cdots, n_k$ pure strategies if there are n_j elements in L_{2j}. Y denotes the set of these strategies.

Random payoffs. A referee, given $x \in X$ and $y \in Y$, co uld play the game, playing the appropriate move from x whenever the game enters one of I's information sets, playing the, appropriate move from y whenever the game enters one of II's information sets, and playing the moves at random with the indicated probabilities at each chance move. The actual outcome of the game for given $x \in X$ and $y \in Y$ depends on the chance moves selected, and is therefore a random quantity. Strictly speaking, random payoffs were not provided for in our definition of games in normal form. However, we are quite used to replacing random payoffs by their average values (expected values) when the randomness is due to the use of mixed strategies by the players. We adopt the same convention in dealing with random payoffs when the randomness is due to the chance moves. The justification of this comes from utility theory

Convention. If for fixed pure strategies of the players, $x \in X$ and $y \in Y$, the payoff is a random quantity, we replace the payoff by the average value, and denote this average value by L(x, y).

For example, if for given strategies $x \in X$ and $y \in Y$, Player I wins 3 with probability 1/4, wins 1 with probability 1/4, and loses 1 with probability 1/2, then his average payoff is $\frac{1}{4}(3)+\frac{1}{4}(1)+\frac{1}{2}(-1)=1/2$ so we let L(x, y) = 1/2.

Therefore, given a game in extensive form, we say (X, Y, L) is the equivalent strategic form of the game if X and Y are the pure strategy spaces of players I and II respectively, and if L(x, y) is the average payoff for $x \in X$ and $y \in Y$.

6 Example. Let us find the equivalent strategic form to Basic Endgame in Poker described in the Section 5.2, whose tree is given in Figure 2. Player I has two information sets. In each set he must make a choice from among two options. He therefore has 2·2 = 4 pure strategies. We may denote them by

(b, b): bet with a winning card or a losing card.

(b, c): bet with a winning card, check with a losing card.

(c, b): check with a winning card, bet with a losing card.

(c, c): check with a winning card or a losing card.

Therefore, X = {(b, b), (b, c), (c, b), (c, c)}. We include in X all pure strategies whether good or bad (in particular, (c, b) seems a rather perverse sort of strategy.)

Player II has only one information set. Therefore, Y = {c, f}, where

c: if I bets, call.

f: if I bets, fold.

Now we find the payoff matrix. Suppose I uses (b, b) and II uses c. Then if I gets a winning card (which happens with probability 1/4), he bets, II calls, and I wins 3 dollars. But if I gets a losing card (which happens with probability 3/4), he bets, II calls, and I loses 3 dollars. I's average or expected winnings is

$$L((b,b),c)=\frac{1}{4}(3)+\frac{3}{4}(-3)=-\frac{3}{2}$$

This gives the upper left entry in the following matrix. The other entries may be computed similarly and are left as exercises.

$$\begin{array}{c} \\ (b,b) \\ (b,c) \\ (c,b) \\ (c,c) \end{array} \begin{array}{c} \begin{array}{cc} c & f \end{array} \\ \begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \\ -2 & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

Let us solve this 4 by 2 game. The third row is dominated by the first row, and the fourth row is dominated by the second row. In terms of the original form of the game, this says something you may already have suspected: that if I gets a winning card, it cannot be good for him to check. By betting he will win at least as much, and maybe more. With the bottom two rows eliminated the matrix becomes $\begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \end{pmatrix}$, whose solution is easily found. The value is V = -1/4. I's optimal strategy is to mix (b, b) and (b, c) with probabilities 1/6 and 5/6 respectively, while II's optimal strategy is to mix c and f with equal probabilities 1/2 each. The strategy (b, b) is Player I's bluffing strategy. Its use entails betting with a losing hand. The strategy (b, c) is Player I's "honest" strategy, bet with a winning hand and check with a losing hand. I's optimal strategy requires some bluffing and some honesty.

In Exercise 4, there are six information sets for I each with two choices. The number of I's pure strategies is therefore $2^6 = 64$. II has 2 information sets each with two choices. Therefore, II has $2^2 = 4$ pure strategies. The game matrix for the equivalent strategic form has dimension 64 by 4. Dominance can help reduce the dimension to a 2 by 3 game! (See Exercise 10(d).)

7 Games of Perfect Information. Now that a game in extensive form has been defined, we may make precise the notion of a game of perfect information.

Definition. A game of perfect information is a game in extensive form in which each information set of every player contains a single vertex.

In a game of perfect information, each player when called upon to make a move knows the exact position in the tree. In particular, each player knows all the past moves of the game including the chance ones. Examples include tic-tac-toe, chess, backgammon, craps, etc.

Games of perfect information have a particularly simple mathematical structure. The main result is that every game of perfect information when reduced to strategic form has a saddle point; both players have optimal pure strategies. Moreover, the saddle point can be found by removing dominated rows and columns. This has an interesting implication for the game of chess for example. Since there are no chance moves, every entry of the game matrix for chess must be either +1 (a win for Player I), or -1 (a win for Player II), or 0 (a draw). A saddle point must be one of these numbers. Thus, either Player I can guarantee himself a win, or Player II can guarantee himself a win, or both players can assure themselves at least a draw. From the game-theoretic viewpoint, chess is a very simple game. One needs only to write down the matrix of the game. If there is a row of all +1's, Player I can win. If there is a column of all -1's, then Player II can win. Otherwise, there is a row with all +1's and 0's and a column with all -1's and 0's, and so the game is drawn with best play. Of course, the real game of chess is so complicated, there is virtually no hope of ever finding an optimal strategy. In fact, it is not yet understood how humans can play the game so well.

8 Behavioral Strategies. For games in extensive form, it is useful to consider a different method of randomization for choosing among pure strategies. All a player really needs to do is to make one choice of an edge for each of his information sets in the game. A behavioral strategy is a strategy that assigns to each information set a probability distributions over the choices of that set.

For example, suppose the first move of a game is the deal of one card from a deck of 52 cards to Player I. After seeing his card, Player I either bets or passes, and then Player II takes some action. Player I has 52 information sets each with 2 choices of action, and so he has 2^{52} pure strategies. Thus, a mixed strategy for I is a vector of 2^{52} components adding to 1. On the other hand, a behavioral strategy for I simply given by the probability of betting for each card he may receive, and so is specified by only 52 numbers.

In general, the dimension of the space of behavioral strategies is much smaller than the dimension of the space of mixed strategies. The question arises – Can we do as well with behavioral strategies as we can with mixed strategies? The answer is we can if both players in the game have perfect recall. The basic theorem, due to Kuhn in 1953 says that in finite games with perfect recall, any distribution over the payoffs achievable by mixed strategies is achievable by behavioral strategies as well.

To see that behavioral strategies are not always sufficient, consider the game of imperfect recall of Figure 4. Upon reducing the game to strategic form, we find the matrix

$$\begin{array}{c} \\ (f,c) \\ (f,d) \\ (g,c) \\ (g,d) \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \end{array}$$

The top and bottom rows may be removed by domination, so it is easy to see that the unique optimal mixed strategies for I and II are (0, 2/3, 1/3, 0) and (2/3, 1/3) respectively. The value is 2/3. However, Player I's optimal strategy is not achievable by behavioral strategies. A behavioral strategy for I is given by two numbers, pf, the probability of choice f in the first information set, and pc, the probability of choice c in the second information set. This leads to the mixed strategy, $(p_f p_c, p_f(1 - p_c), (1 - p_f)p_c, (1 - p_f)(1 - p_c))$. The strategy (0, 2/3, 1/3, 0) is not of this form since if the first component is zero, that is if $p_f p_c = 0$, then either $p_f = 0$ or $p_c = 0$, so that either the second or third component must be zero also.

If the rules of the game require players to use behavioral strategies, as is the case for certain models of bridge, then the game may not have a value. This means that if Player I is required to announce his behavioral strategy first, then he is at a distinct disadvantage. The game of Figure 4 is an example of this. (see Exercise 11.)

9 Exercises.

1. Player II chooses one of two rooms in which to hide a silver dollar. Then, Player I, not knowing which room contains the dollar, selects one of the rooms to search. However, the search is not always successful. In fact, if the dollar is in room #1 and I searches there, then (by a chance move) he has only probability 1/2 of finding it, and if the dollar is in room #2 and I searches there, then he has only probability 1/3 of finding it. Of course, if he searches the wrong room, he certainly won't find it. If he does find the coin, he keeps it; otherwise the dollar is returned to Player II. Draw the game tree.

2. Draw the game tree for problem 1, if when I is unsuccessful in his first attempt to find the dollar, he is given a second chance to choose a room and search for it with the same probabilities of success, independent of his previous search. (Player II does not get to hide the dollar again.)

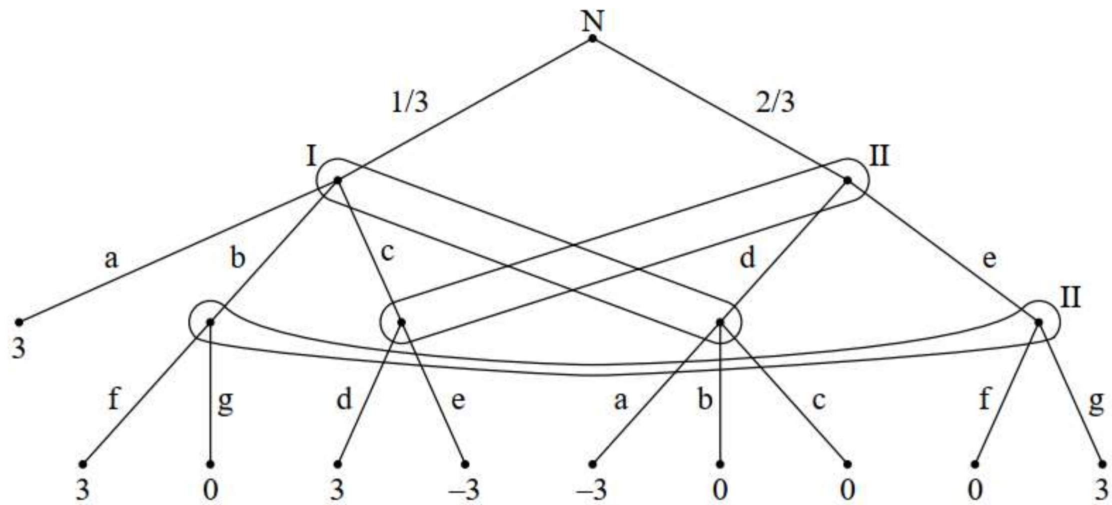
3. A Statistical Game. Player I has two coins. One is fair (probability 1/2 of heads and 1/2 of tails) and the other is biased with probability 1/3 of heads and 2/3 of tails. Player I knows which coin is fair and which is biased. He selects one of the coins and tosses it. The outcome of the toss is announced to II. Then II must guess whether I chose the fair or biased coin. If II is correct there is no payoff. If II is incorrect, she loses 1. Draw the game tree.

4. A Forgetful Player. A fair coin (probability 1/2 of heads and 1/2 of tails) is tossed and the outcome is shown to Player I. On the basis of the outcome of this toss, I decides whether to bet 1 or 2. Then Player II hearing the amount bet but not knowing the outcome of the toss, must guess whether the coin was heads or tails. Finally, player I (or, more realistically, his partner), remembering the amount bet and II's guess, but not remembering the outcome of the toss, may double or pass. II wins if her guess is correct and loses if her guess is incorrect. The absolute value of the amount won is [the amount bet (+1 if the coin comes up heads)] (×2 if I doubled). Draw the game tree.

5. The Kuhn Poker Model. Two players are both dealt one card at random from a deck of three cards {1, 2, 3}. (There are six possible equally likely outcomes of this chance move.) Then Player I checks or bets. If I bets, II may call or fold. If I checks, II may check or bet. If I checks and II bets, then I may call or fold. If both players check, the player with the higher card wins 1. If one player bets and the other folds, the player who bet wins 1. If one player bets and the other calls, the player with the higher card wins 2. Draw the game tree. (H. W. Kuhn, "A simplified two-person poker" Contributions to the Theory of Games, vol. I, pg. 97, 1950, Ed. Kuhn and Tucker, Princeton University Press.)

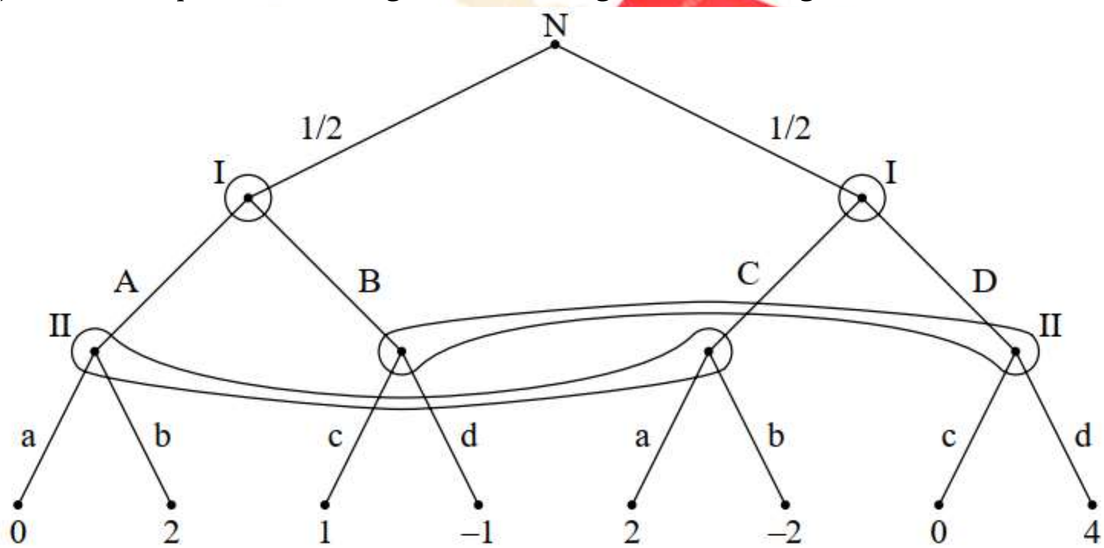
6. Generalize Basic Endgame in poker by letting the probability of receiving a winning card be an arbitrary number p, $0 \leq p \leq 1$, and by letting the bet size be an arbitrary number $b > 0$. (In Figure 2, 1/4 is replaced by p and 3/4 is replaced by 1 - p. Also 3 is replaced by 1 + b and -3 is replaced by -(1 + b).) Find the value and optimal strategies. (Be careful. For $p \geq (2 + b)/(2 + 2b)$ there is a saddle point. When you are finished, note that for $p < (2 + b)/(2 + 2b)$, Player II's optimal strategy does not depend on p!)

7. (a) Find the equivalent strategic form of the game with the game tree:



(b) Solve the game.

8. (a). Find the equivalent strategic form of the game with the game tree:



(b). Solve the game.

9. Coin A has probability 1/2 of heads and 1/2 of tails. Coin B has probability 1/3 of heads and 2/3 of tails. Player I must predict "heads" or "tails". If he predicts heads, coin A is tossed. If he predicts tails, coin B is tossed. Player II is informed as to whether I's

prediction was right or wrong (but she is not informed of the prediction or the coin that was used), and then must guess whether coin A or coin B was used. If II guesses correctly she wins 1 dollar from I. If II guesses incorrectly and I's prediction was right, I wins 2 dollars from II. If both are wrong there is no payoff.

(a) Draw the game tree.

(b) Find the equivalent strategic form of the game.

(c) Solve.

10. Find the equivalent strategic form and solve the game of

(a) Exercise 1.

(b) Exercise 2.

(c) Exercise 3.

(d) Exercise 4.

11. Suppose, in the game of Figure 4, that Player I is required to use behavioral strategies. Show that if Player I is required to announce his behavioral strategy first, he can only achieve a lower value of 1/2. Whereas, if Player II is required to announce her strategy first, Player I has a behavioral strategy reply that achieves the upper value of 2/3 at least.

12. (Beasley (1990), Chap. 6.) Player I draws a card at random from a full deck of 52 cards. After looking at the card, he bets either 1 or 5 that the card he drew is a face card (king, queen or jack, probability 3/13). Then Player II either concedes or doubles. If she concedes, she pays I the amount bet (no matter what the card was). If she doubles, the card is shown to her, and Player I wins twice his bet if the card is a face card, and loses twice his bet otherwise.

(a) Draw the game tree. (You may argue first that Player I always bets 5 with a face card and Player II always doubles if Player I bets 1.)

(b) Find the equivalent normal form.

(c) Solve.

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